SINGLE FACTOR MODEL

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Single Factor Model: (Developed by Vasicek, similar in thinking to CAPM) also called single factor Gaussian copula model. The credit index corresponding to a given name as a systematic component plus an idiosyncratic component

$$Y_n = \beta_n Z + \sqrt{1-\beta_n^2} \cdot \epsilon_n$$

where Z and ϵ_n are i.i.d. standard normal and the default indicator has the form

$$\mathbf{1}_{\{Y_n \leq H_n\}}$$

Clearly it is a Bernoulli RV with mean PD_n and variance $PD_n \cdot (1 - PD_n)$.

- The correlation between the credit indices of two different names is
$$\rho_{Y_n,Y_m} = \frac{cov\left(Y_n, Y_m\right)}{\sigma_{Y_n}\sigma_{Y_m}} = \mathbb{E}\left[Y_n Y_m\right] - \mathbb{E}\left[Y_n\right]\mathbb{E}\left[Y_m\right] = \beta_n\beta_m\mathbb{E}\left[Z^2\right] = \beta_n\beta_m$$
- The correlation between the default indicators of two different names is
$$\rho = \frac{\mathbb{E}\left[\mathbf{1}_{\{Y_n\leq H_n\}} \cdot \mathbf{1}_{\{Y_m\leq H_m\}}\right] - PD_n \cdot PD_m}{\sqrt{PD_n \cdot (1-PD_n) \cdot PD_m \cdot (1-PD_m)}}$$
- Bivariate normal distribution with correlation r is given by
$$\mathcal{N}_2\left(x, y, r\right) = \mathbb{P}\left(X \leq x, Y \leq y\right) = \int_{-\infty}^{x}\int_{-\infty}^{y} e^{-\frac{x^2-2rxy+y^2}{2\left(1-r^2\right)}} \frac{dxdy}{2\pi\sqrt{1-r^2}}$$
- Total portfolio loss is given by
$$L_T = \sum_{n=1}^{N} w_n\left(1 - RR_n\right) \cdot \mathbf{1}_{\{Y_n\leq H_n\}}$$
and the systematic loss is given by
$$L_S = \mathbb{E}\left[L_T | Z\right]$$
which is often referred to as the "large portfolio limit".
- Conditional expectation for a given name is deduced as follows
$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{Y_n\leq H_n\}} | Z\right] &= \mathbb{P}\left[Y_n \leq H_n | Z\right] \\
&= \mathbb{P}\left[\beta_n Z + \sqrt{1-\beta_n^2} \cdot \epsilon_n \leq H_n | Z\right] \\
&= \mathbb{P}\left[\left[\epsilon_n \leq \frac{H_n - \beta_n Z}{\sqrt{1-\beta_n^2}} | Z\right]\right] \\
&= \mathcal{N}\left(\frac{\mathcal{N}^{-1}\left(PD_n\right) - \beta_n Z}{\sqrt{1-\beta_n^2}}\right)
\end{aligned}$$

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Hence the systematic loss is a decreasing function of the global systematic factor Z by observing

$$L_S = \sum_{n=1}^{N} w_n \left(1 - RR_n\right) \mathcal{N}\left(\frac{\mathcal{N}^{-1}\left(PD_n\right) - \beta_n Z}{\sqrt{1-\beta_n^2}}\right)$$

- The VaR $L_{S,\alpha}$ is simply given by

$$L_{S,\alpha} = \sum_{n=1}^{N} w_n \left(1 - RR_n\right) \mathcal{N}\left(\frac{\mathcal{N}^{-1}\left(PD_n\right) - \beta_n Z_{1-\alpha}}{\sqrt{1-\beta_n^2}}\right)$$

by applying **Euler's Theorem**

$$L_{S,\alpha} = \sum_{n=1}^{N} w_n \frac{\partial L_{S,\alpha}}{\partial w_n}$$

where

$$\frac{\partial L_{S,\alpha}}{\partial w_n} = \left(1 - RR_n\right) \mathcal{N}\left(\frac{\mathcal{N}^{-1}\left(PD_n\right) - \beta_n Z_{1-\alpha}}{\sqrt{1-\beta_n^2}}\right)$$

Basel Accord II: The single factor model forms the basis for the credit risk approach of the second Basel Capital Accord. The economic capital is based on VaR which is at the 99.9% confidence level with a one year time horizon.

The capital charge for an instrument is

$$EAD \cdot LGD \cdot (WCDR - PD) \cdot MA$$

where

$$WCDR = \mathcal{N}\left(\frac{\mathcal{N}^{-1}\left(PD_n\right) - \beta_n Z_{0.001}}{\sqrt{1-\beta_n^2}}\right)$$